Journal of Algebra 321 (2009) 117–127

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Infinitesimal Hecke algebras of $\mathfrak{sl}_2$ in positive characteristic

Akaki Tikaradze

The University of Toledo, Department of Mathematics, Toledo, OH, USA

ARTICLE INFO

Article history:
Received 24 November 2007
Available online 31 October 2008
Communicated by Vera Serganova

Keywords:
Representation theory
Quantum algebras

ABSTRACT

We describe centers of infinitesimal Hecke algebra of $\mathfrak{sl}_2$ in positive characteristic. In particular, we show that these algebras are finitely generated modules over their centers, and the Azumaya and smooth loci of the centers coincide.

1. Introduction

In this paper, we will work over an algebraically closed field k of characteristic $p > 2$. In this setting, we will consider infinitesimal Hecke algebras associated to the Lie algebra $\mathfrak{sl}_2$ and its natural representation (see [EGG] for the definition of infinitesimal Hecke algebras associated to an arbitrary reductive Lie algebra and its representation). We will be concerned with their centers and finite-dimensional representations. Let us recall the precise definition of these algebras.

Let V be the natural 2-dimensional representation of $\mathfrak{g} = \mathfrak{sl}_2$. Let us fix its basis elements x, y so that $ex = 0$, $fx = y$, $hx = x$, $hy = -y$, here e, f, h denote the standard basis elements of $\mathfrak{sl}_2$. Then for any $z \in k[\Delta]$, where Δ denotes the rescaled Casimir element $h^2 + 4ef - 2h$, one defines an algebra H_z as the quotient of $\mathfrak{U}\mathfrak{g} \ltimes TV$ by the two sided ideal generated by the element $[x, y] - z$, where TV denotes the tensor algebra of V. Thus we get a family of algebras parametrized by elements in $k[\Delta]$. The algebra H_z can be equipped with a natural algebra filtration $F^n H_z$ such that $\mathfrak{U}\mathfrak{g} = F^0 H_z$, $\mathfrak{U}\mathfrak{g}V = F^1 H_z$ and $F^n H_z = (F^1 H_z)^n$. The main property of H_z is that it satisfies the PBW property, namely the natural surjection $H_0 = \mathfrak{U}(\mathfrak{g} \ltimes V) \to \operatorname{gr}(H_z)$ is an isomorphism [Kh,EGG]. Thus one can think of algebras H_z as some kind of deformations of the enveloping algebra $H = H_0$ depending on a deformation parameter $z \in k[\Delta]$.

When the ground field k has characteristic 0, we proved in [Ti] that the center of H_z is isomorphic to the polynomial ring in one variable and $\operatorname{gr}(\mathfrak{Z}(H_z)) = \mathfrak{Z}(\operatorname{gr}(H_z)) = \mathfrak{Z}(H)$ (where the letter $\mathfrak{Z}$ stands for the center of an algebra). In this paper we show that if $\deg(z) < p - 1$, then al-

E-mail address: tikar@math.uchicago.edu.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.10.005

gebra H_z is a finitely generated module over its center and just like in the characteristic 0 case $\mathrm{gr}(\mathfrak{Z}(H_z)) = \mathfrak{Z}(\mathrm{gr}(H_z)) = \mathfrak{Z}(H)$. In this way we have a similar picture to the case of symplectic reflection algebras [BGF]. Geometrically $\operatorname{Spec}\mathfrak{Z}(H_z)$ turns out to be a p-fold ramified cover of the affine space $\mathbb{A}^5$. At the end of the paper, we discuss irreducible modules of H_z. In particular, we prove that the Azumaya and the smooth loci of $\mathfrak{Z}(H_z)$ coincide. Analogous statements are known to be true for Cherednik algebras [BC], enveloping algebras of semi-simple Lie algebras and quantized enveloping algebras [BG]. In the special case of $z = 0$, we describe all irreducible modules of H.

2. Center and irreducible representations

Throughout this paper, we will use the term 'maximal vector' for elements of representations of $\mathfrak{g}$ annihilated by e. Let us recall the following computation from [Ti] which we are going to use later. For any $\omega \in k[\Delta]$ we have

$$[\omega, x] = \big(F(\omega)h + G(\omega)\big)x + 2eF(\omega)y,$$

where F, $G : k[\Delta] \to k[\Delta]$ are linear endomorphisms of $k[\Delta]$ defined recursively as follows:

$$F\big(\Delta^{n+1}\big) = 2\Delta^n + (\Delta - 1)F\big(\Delta^n\big) - 2G\big(\Delta^n\big),$$
$$G\big(\Delta^{n+1}\big) = -3\Delta^n + (\Delta + 3)G\big(\Delta^n\big) - 2\Delta F\big(\Delta^n\big).$$

It is immediate that $\deg F(\Delta^n) = \deg G(\Delta^n) = n - 1$ and leading coefficient of $F(\Delta^n)$ is $2n$, and the leading coefficient of $G(\Delta^n)$ is $-n(2n + 1)$. When $\mathrm{char}(k) = 0$, it was shown in [Ti] that center of H_z is generated by the element $t_z = ex^2 + hxy - fx^2 - \frac{1}{2}hz - \omega_z$, where $\omega_z = -F^{-1}(z) + \frac{1}{2}z + \frac{1}{2}F^{-1}(G(z))$. We see from this formula that the element t_z can be defined in positive characteristic setting as long as $\mathrm{char}(k) < p - 1$.

We will also use the following anti-involution j of H_z defined as follows:

$$j(x) = y, \qquad j(y) = x, \qquad j(h) = h, \qquad j(e) = -f, \qquad j(f) = -e.$$

Main result of the paper is the following

Theorem 2.1. *If $\deg z < p - 1$, then the center of H_z is generated as an algebra over k by e^p, f^p, $h^p - h$, x_p, y_p, t_z, where x_p has top symbol with respect to the filtration equal to x^p, and y_p has y^p. $\operatorname{Spec}(\mathfrak{Z}(H_z))$ is a finite (ramified) cover of $\mathbb{A}^5 = \operatorname{Spec}(k[e^p, f^p, h^p - h, x_p, y_p])$ of degree p.*

The proof will be divided into several steps. As a first step, we will show that the center of H (the associated graded of H_z) is generated by e^p, $h^p - h$, f^p, x^p, y^p, t (where $t = t_0 = ex^2 + hxy - fx^2$).

We will argue by induction on the filtration degree of the central element. The first thing that needs to be checked is that $\mathfrak{Z}(H) \cap \mathfrak{U}\mathfrak{g} = k[e^p, f^p, h^p - h]$. Since $\mathfrak{Z}(\mathfrak{U}\mathfrak{g}) = k[e^p, f^p, h^p - h, \Delta]$ [V], any $a \in \mathfrak{Z}(H) \cap \mathfrak{U}\mathfrak{g}$ may be expressed as $a = \sum_{i<p} \alpha_i \Delta^i$, where $\alpha_i \in k[e^p, f^p, h^p - h]$. We have

$$0 = [a, x] = \sum \alpha_i\big[\Delta^i, x\big],$$

thus $\sum \alpha_i F(\Delta^i) = 0$ (recall that F is the linear endomorphism of $k[\Delta]$ from the beginning.) Since $F(\Delta^i)$ has a degree $i - 1$ (with leading coefficient $2i$), we get that $\alpha_i = 0$ for $i > 0$ since elements $1, \Delta, \ldots, \Delta^{p-1}$ are linearly independent over $k[h^p - h, e^p, f^p]$, therefore $a \in k[h^p - h, f^p, e^p]$.

Now, we will deal with central elements of H of positive degree in x, y. Let us begin by making some preliminary remarks about homogeneous elements of H that commute with e and y. Let a be such an element. We may write

$$a = \left(\sum_{i=0}^{n} b_i y^i x^{n-i}\right) x^m, \quad b_i \in \mathfrak{U}\mathfrak{g},\ b_n \neq 0.$$

From $[e, a] = 0$ we get that $ib_i = -[e, b_{i-1}]$. In particular, $[e, b_n] = 0$. Also from $[a, y] = 0$ we get that $[b_n, y]$ contains no elements from $\mathfrak{U}\mathfrak{g}y$ in its PBW monomial expression. Now we have the following

Lemma 2.1. *Let $\alpha \in \mathfrak{U}\mathfrak{g}$ be an element which commutes with e, such that $[\alpha, y]$ has no terms from $\mathfrak{U}\mathfrak{g}y$ in its PBW monomial expansion. Then α belongs to $k[h^p - h, f^p, e]$.*

Proof. We will argue by induction on the filtration degree (with respect to the standard filtration of $\mathfrak{U}\mathfrak{g}$). Let $\alpha \in \mathfrak{U}\mathfrak{g}$ be an element which satisfies the conditions of the lemma. Let us write

$$\alpha = \sum f^i \alpha_i, \quad \alpha_i \in k[h, e]$$

exactly in this order, meaning that h is on the left, e is on the right. Then $[\alpha, y]$ has no y if and only if $[\alpha_i, y]$ has no y. Let

$$\beta = \sum a_{ij} h^i e^j, \quad a_{ij} \in k$$

be such that $[\beta, y]$ has no y. Then

$$[\beta, y] = \sum j a_{ij} h^i e^{j-1} x + \sum a_{ij} \big[h^i, y\big] e^j,$$

and $[h^i, y] = (-ih^{i-1} + \text{lower powers of } h)y$, therefore $[h^i, y]e^j = -ih^{i-1}e^j y + \text{terms with lower powers of } h$. Hence the highest i with $a_{ij} \neq 0$ is a multiple of p. Then considering the element $\beta - (h^p - h)^{i/p} a_{ij} e^j$, we see that β may be written as a linear combination of elements of the form $(h^p - h)^i e^j$. Therefore

$$\alpha = \sum_{i=0}^{m} \big(h^p - h\big)^i \beta_i, \quad \beta_i \in k[f, e],$$

where f is on the left from e. Then

$$[e, \alpha] = 0 = \sum \big(h^p - h\big)^i [e, \beta_i].$$

This implies that $[e, \beta_k] = 0$. Indeed, looking at the terms with the highest power of h, it will equal to h^{pk} (term with the highest power of h in $[e, \beta_k]$) (where we right h on the left in the PBW expression of $[e, \beta_k]$). Thus, each β_i commutes with e. Let us write

$$\beta_i = \sum_{j=0}^{m} f^j g_j, \quad g_j \in k[e].$$

Assume without loss of generality that $m < p$. Now $[e, \beta_i]$ will contain $hf^{m-1} g_m$, and this forces $m = 0$. $\square$

Applying the anti-involution j, we get an analogous result for f and x.

Let a as above be a homogeneous central element, we claim that m is a multiple of p. Indeed, from $0=[f,a]$, looking at highest power of y, it is clear that $b_n y^{n+1} m x^{m-1}=0$. hence p divides m, so without loss of generality we may assume that $m=0$.

Let us write $n=pl+m$ (not to be confused with the old m), $0\leqslant m<p$. From the above lemma, we conclude that $b_n=\alpha e^k, b_0=\alpha' f^k$ for some $\alpha,\alpha'\in k[e^p, f^p, h^p-h], k<p$. We also have $(\operatorname{ad} e)^{m+1} f^k=0$, but $m=2k \bmod p$ (since a has a weight 0 with respect to h), therefore either $m=2k$, or $m=2k-p$. The latter case is impossible since then $m<k$, which may not happen by the Lemma below. Thus $m=2k$. Now consider the element $a-t^m\alpha(x^p)^l$. This is a central element divisible by y. Hence, proceeding by induction, we are done.

The following is well known [St], but we include the proof for the sake of completeness.

Lemma 2.2. *If $k<p$, then $(\operatorname{ad} e)^k(f^k)\neq 0$.*

Proof. If we write $(\operatorname{ad} e)^k(f^k)$ as a sum of monomials in the PBW basis h,e,f then each term will have a degree at most k, and it will have a term $k!h^k$ in it. Therefore it cannot be 0. $\square$

For the sake of brevity, let us denote the ring $k[e^p, h^p-h, f^p]$ by R. So far we have proved that $\mathfrak{Z}(H)=R[x^p, y^p, t]$. Now we turn to the case of a nonzero parameter z. Using the well-known identity $(\operatorname{ad} a)^p=\operatorname{ad}(a^p)$, we see that elements e^p, h^p-h, f^p, t_z lie in the center of H_z. We have a natural injection $\operatorname{gr}(\mathfrak{Z}(H_z))\to\mathfrak{Z}(H)$ and we have to show that it is actually an isomorphism, so it remains to demonstrate the existence of $x_p, y_p\in\mathfrak{Z}(H_z)$ such that they map to x^p, y^p respectively. By virtue of the anti-involution j, it will suffice to prove the existence of x_p. We will prove this by analyzing maximal vectors in H (with respect to the adjoint action of $\operatorname{ad}(\mathfrak{g})$) and manipulating the anti-involution j. In a sense, the proof is similar in spirit to the proof of the existence of t_z in characteristic 0 [Ti].

Let us start by establishing certain facts about $\mathfrak{U}\mathfrak{g}$ which will be used in a crucial way later. (These facts may be very well known but we could not find them in the literature.)

Lemma 2.3. *Let $a\in\mathfrak{U}\mathfrak{g}$ be a maximal vector. Then a lies in $R[\Delta,e]$*

Proof. We may assume that a is homogeneous with respect to $\operatorname{ad}(h)$, say of weight n and is not divisible by e from the left. Let us write a as a sum of PBW monomials

$$a=\sum_{i=0}^{m} e^i g_i(h) f^j,\quad g_i(h)\in k[h],\ g_0(h), g_m(h)\neq 0,\ 2(i-j)=n \bmod(p).$$

From $[e,a]=0$ we get that $[g_m(h),e]$ has no terms beginning with e and $[e, f^{-\frac{n}{2}\bmod(p)}]=0$. Thus, $n=0$ and $g_m(h)\in R$. Hence, if we consider $g_m(h)(\frac{1}{4}\Delta)^m-a$ and apply induction on m we will be done. $\square$

Proposition 2.1. *Let $\alpha\in\mathfrak{Z}(\mathfrak{U}\mathfrak{g})$ such that it has no terms involving e^p. If $\alpha f^m=[e,b]$ for $0<k<p$ and $b\in\mathfrak{U}\mathfrak{g}$, then $\alpha=0$.*

Proof. First, suppose α is divisible by f^p. Then we may write $\alpha=\beta f^p$. If we write b as sum of PBW monomials in $e^i h^j f^k$, taking the commutator of each such monomial with e of never increases power of f in it. This implies that some of monomials of b with the power of f less than p must commute with e, hence we may disregard it. Thus we may write $b=b' f^p$. Therefore without loss of generality we may assume that α is not divisible by f^p.

We claim that in this case there exists a regular semi-simple character χ such that α will not vanish on V_λ. Let us recall the corresponding definitions [FP]. A character is called regular semi-simple if it lies in the coadjoint orbit under the action of $SL_2(k)$ of a character defined as follows:

$$\chi(e^p)=\chi(f^p)=0,\qquad \chi(h^p-h)=c=\lambda^p-\lambda,\quad c\neq 0\in k,$$

and

$$V_{\chi,\lambda} = \mathfrak{U}\mathfrak{g}_\chi \otimes_B k_\lambda$$

where B is a subalgebra generated by e, h and k_λ is its one-dimensional representation on which e vanishes and h acts as a multiplication by λ where $\lambda^p - \lambda = c$. Now the desired statement boils down to the following statement about polynomials in one variable. We thank M. Boyarchenko for the following quick proof.

Lemma 2.4. *If $\sum g_i(s(s+2))(s^p - s)^i = 0$ for $g_i(s) \in k[s]$, $\deg(g_i) < p$ then all g_i must equal 0.*

Proof.
If we make the substitution $s = s - 1$ and replace $g_i(s)$ by $g_i(s-1)$, then we will have

$$\sum_0^m g_i(s^2)(s^p - s)^i = 0, \quad g_m \neq 0, \ g_0 \neq 0.$$

We have

$$0 = \sum s^i g_i(s^2)\left(s^{\frac{p-1}{2}} - 1\right)^i.$$

Then, looking at even powers of s we conclude that

$$0 = \sum_i g_{2j}(s^2)\left((s^p - s)^2\right)^j.$$

Since $\deg(g_{2j}(s^2)) < 2p$, we get that all $g_{2j} = 0$, a contradiction. □

As it is well known [FP], these modules are irreducible and have as a basis $v_\lambda, f v_\lambda, \ldots, f^{p-1} v_\lambda$. To complete the proof of the proposition, it will suffice to check that $f^m \in \mathrm{ad}(e)(End(V_\lambda))$ only for finitely many $\lambda \in k$. This is very easy to see since if we have $f^m = [e, A]$, where $A \in End(V_\lambda)$, then $A = a_i : kf^i v_\lambda \to kf^{i+m+1} v_\lambda$, where $a_i \in k$. If we write out the commutator condition we immediately see that λ is a root of a polynomial in F_p which is independent of λ, hence there are only finitely many such λ. □

Our goal will be to produce an element A_x of weight 0 such that $A_x \in R[t_z, e, x] \cap F^{p-1} H_z$ and $[A_x, \Delta] = [x^p, \Delta]$. By the lemma below, this will complete the proof.

Lemma 2.5. *Let $A_x \in R[t_z, e, x]$ be an element of weight 0 lying in $F^{p-1} H_z$, such that $[x^p - A_x, \Delta] = 0$. Then $x^p - A_x$ lies in the center of H_z.*

Proof. Since $x^p - Ax$ commutes with e, h we get that $[x^p - A_x, f] = 0$. Applying $\mathrm{ad}\, x$ to this, we get that $[x^p - A_x, y] = 0$, hence $x^p - A_x$ is a central element. □

The following describes maximal vectors in $F^{p-1} H$.

Proposition 2.2. *Let $A \in F^{p-1} H$ be a homogeneous (in x, y) maximal vector. Then it can be expressed as a sum $B + C$, where B is a linear combination of elements of the form $\gamma e^j [x^i, \Delta]$, $\gamma e^j x^i$, $\gamma \in R[t]$, while C is a homogeneous maximal vector whose coefficient of highest power of y lies in $\mathfrak{Z}(\mathfrak{U}\mathfrak{g})$ and is not divisible by e.*

Proof. First we establish that $[\Delta, x]x^n$ can be expressed as a linear combination of elements of the form $\gamma e^j[x^i, \Delta], \gamma e^j x^i$. Indeed,

$$\begin{aligned}\big[\Delta, x^n\big] &= \sum_{k<n} x^k\big((2h-3)x + 4ey\big)x^{n-k-1} \\ &= n\big((2h-3) + 4ey\big)x^{n-1} + c'x^n \\ &= n[\Delta, x]x^{n-1} + c'x^n,\end{aligned}$$

where c' is some constant. Thus B can be taken from the span of $\gamma e^i[x, \Delta]x^j$, $\gamma e^i x^j$ which is a $k[f^p, h^p - h, \Delta, t] - k[x]$-module. Let A be a homogeneous (with respect to $\operatorname{ad} h$ and in x, y) maximal vector of degree n. Without loss of generality we may assume that it is not divisible by x from the right, so we may write $A = \sum a_i x^i y^{n-i}$ with $a_0 \neq 0$. We know that a_0 must be a maximal vector, thus by the above lemma we may write $a_0 = \alpha e^m$, where $\alpha \in R[\Delta]$ and it is not divisible by e. Now, if $\frac{1}{2}n \leqslant m$, then considering one of the following elements

$$\alpha e^{m-\frac{1}{2}n}t^{\frac{1}{2}n} - A, \qquad \alpha e^{m-\frac{1}{2}(n-1)}t^{\frac{1}{2}(n-1)}\frac{1}{4}[\Delta, x] - A,$$

it will be divisible by x from the right and we may apply induction on the degree of A.

So it just remains to deal with the case $2m < n$. In this case $\alpha e^m = c(\operatorname{ad} e)^n(a_n)$ (here we also need the fact that A cannot be divisible by y from the right, which was established earlier in the proof), where c is some nonzero constant. Applying $(\operatorname{ad} f)^{2m}$ to both sides we get that $\alpha f^m = [e, b]$ for some $b \in \mathfrak{U}\mathfrak{g}$, but as the above proposition shows this can only happen when $m = 0$, in which case we get an element of type C from the statement of the proposition. $\square$

Passing to the associated graded, we see that we may replace H by H_z in the lemma above. Now let us look at $[X^p, \Delta]$. Clearly this is a maximal vector lying in $F^{p-1}H_z$. So we may apply the above proposition. Using an analogous proposition for y, we may write

$$\big[x^p, \Delta\big] = [A_x, \Delta] + B_x + C,$$

where $A_x, B_x \in R[t_z, \Delta, e, x]$ and C is a maximal vector whose coefficient in front of the highest power of y is not divisible by e. However $[x^p, \Delta]$ is divisible by e from the left and so are all the terms $[A_x, \Delta], B_x$ (since they have weight 0), forcing $C = 0$. Analogously for y, we have $[y^p, \Delta] = [A'_y, \Delta] + B'_y$, where $A'_y, B'_y \in R[t_z, \Delta, f, y]$ all have weight 0 (with respect to $\operatorname{ad}(h)$ as always). Our objective is to prove that $B_x = 0$ and $A_x \in R[e, x, t_z]$. We will accomplish this by utilizing the involution j. If we apply it to the last equality we will get

$$\big[x^p, \Delta\big] = -\big[j\big(y^p\big), j(\Delta)\big] = \big[j(A'_y), \Delta\big] - j\big(B'_y\big) = [a_x, \Delta] + B_x.$$

But on the other hand

$$\begin{aligned}\big[x^p, \Delta\big] &= 4e\big[x^p, f\big] = -4e(\operatorname{ad} x)^{p-2}(z), \\ \big[y^p, \Delta\big] &= 4f\big[y^p, e\big] = 4f(\operatorname{ad} y)^{p-2}(z).\end{aligned}$$

Therefore,

$$\begin{aligned}e^{-1}\big[x^p, \Delta\big] &= -\frac{1}{(p-2)!}(\operatorname{ad} e)^{p-2}\big(f^{-1}\big[y^p, \Delta\big]\big) \\ &= -(\operatorname{ad} e)^{p-2}\big(f^{-1}\big[y^p, \Delta\big]\big).\end{aligned}$$

Comparing above formulas, we get

$$\left[j(A'_y), \Delta\right] - j(B'_y) = -e\left(\left[ad(e)^{p-2}\left[f^{-1}A'_y, \Delta\right] + \operatorname{ad}(e)^{p-2}f^{-1}B'_y\right]\right).$$

Now if $B'_y = f^i y^{p-i}$, then

$$\begin{aligned}(\operatorname{ad} e)^{p-2}\left(f^{i-1}y^{p-2i}\right) &= (-1)^{i-1}(p-2)!e^{i-1}x^{p-2i} \\ &= (-1)^{i-1}e^{i-1}x^{p-2i},\end{aligned}$$

so

$$-j\left(f^i y^{p-2i}\right) = (-1)^i e^i x^{p-2i} = (\operatorname{ad} e)^{p-2}\left(B'_y\right).$$

Thus if we write

$$A'_y = \alpha_i\phi_i(\Delta)y^i, \qquad B'_y = \beta_i\psi_i(\Delta)y^i$$

where $\alpha_i, \beta_i \in R[f, t_z]$; $\phi_i(\Delta), \psi_i(\Delta) \in \Delta k[\Delta]$, we will get that

$$\left[\alpha_i\left[\phi_i(\Delta), y^i\right] + \beta_i\psi_i(\Delta)y^i, \Delta\right] = 2B'_y.$$

The next couple of lemmas imply that from the last equality it follows that $B'_y = 0$ and ϕ_i are constants (they do not contain Δ) which will end the proof of existence of x_p.

Lemma 2.6. *Let $A \in R[\Delta, e, x, t_z] \cap F^{p-1}H_z$ be of weight 0. Then there does not exist a maximal vector B which can be expressed as a sum of elements of the form $\alpha_i t_z^i[x, \Delta]x^j$, $\alpha_i \in R[\Delta, e]$, such that $[B, \Delta] = A$.*

Proof. By passing to the associated graded, it suffices to consider the case $z = 0$. Assume that such a B exists. We may assume that A is homogeneous in x, y, since $[B, \Delta] = 4e[B, f]$. By dividing by e from the left we get $[B, f] = A_1$ with $A_1 \in R[\Delta, e, t]$, $\operatorname{wt}(A) = -2$. Let us write $B = (\sum a_i x^i y^{n-i})x^k$, where $a_0 \neq 0$. After picking terms with the lowest powers of x we get

$$ka_0 y^n x^{k-1} = \beta f^r y^{2r} x^{n-2r}$$

for some r, where $\beta \in k[\Delta, e]$ implying that f^r is a maximal vector. This is false unless $r = 0$, thus $k = 0$. Writing B as a sum of elements of type $\alpha_i t^i[x, \Delta]x^{n-2i-1}$, we see by the preceding remark that it must contain a nonzero term of the form $\alpha t^{n-1/2}[x, \Delta]$ with $\alpha \in R[\Delta]$. In particular, n must be odd, hence A may not contain any terms with no powers of x in it. If we can show that $[B, f]$ has such a term, we will have a contradiction. We have $\alpha t^{n-1/2}[[\Delta, x], f] = t^{n-1/2}[\alpha[\Delta, x], f]$, since weight of $B = 0$, $\alpha = \gamma e^m$, where $m = \frac{1}{2}(p-1)$, $\gamma \in R[\Delta]$. Hence it would suffice to show that $[e^k[\Delta, x], f]$ contains y. We have

$$\left[e^k[\Delta, x], f\right] = -e^k[\Delta, y] + \left[e^k, f\right][\Delta, x].$$

Grouping y terms in the above we get the following term

$$\left(\left[e^k, f\right]4e - e^k(2h+3)\right)y.$$

Straightforward computation shows that this term is not 0. $\square$

Lemma 2.7. *If an element $A \in F^{p-1}H_z$ commutes with $\mathfrak{g}$, then it belongs to $R[\Delta, t_z]$.*

Proof. We may pass to the associated graded, so once again we may assume that $z = 0$. As explained before A may not be divisible by either x, y. Assuming that it is homogeneous of degree n, we may write

$$A = \sum \alpha_i x^{n-i} y^i, \quad \alpha_0,\ \alpha_n \neq 0,\ \alpha_0 = \alpha e^m,\ \alpha_n = \beta e^m$$

where $\alpha, \beta \in R[\Delta]$ and $n = 2m$. Hence $A - \alpha t^m$ is divisible by x, so $A = \alpha t^m$. $\square$

Lemma 2.8. *Let $A = \sum_i \alpha_i t^i [\phi_i, x^{n-2i}]$, $\alpha_i \in R[e]$ be a homogeneous element (in x, y) of weight 0 such that $\phi_i \in \Delta k[\Delta]$. Then $[A, \Delta] = 0$ if and only if all α_i are 0.*

Proof. Since A commutes with e and h, we get that it must lie in the centralizer of $\mathfrak{g}$, hence by the previous lemma we have

$$A = \sum_i \alpha_i t^i \big[\phi_i, x^{n-2i}\big] = \beta t^{\frac{n}{2}}, \quad \beta \in R[\Delta].$$

This equality implies in particular that $\alpha_0[\phi_0, x^n]$ is divisible by t. Let us show that this cannot be the case. Consider the universal enveloping algebra filtration on H (recall that $H = \mathfrak{U}(\mathfrak{sl}_2 \ltimes V)$). The top symbol of $\alpha_0[\phi_0, x^n]$ with respect to this filtration is equal to

$$n\big(F(\phi_0)h + G(\phi_0)x + 2eF(\phi_0)y\big)x^{n-1},$$

which is clearly not divisible by $t = ey^2 + hxy - fx^2$ in $\mathrm{gr}(H) = \mathrm{Sym}(\mathfrak{sl}_2 \oplus V)$, hence $\alpha_0[\phi_0, x^n]$ is not divisible by t in H. $\square$

Next, we will show that elements $1, t, \ldots, t^{p-1}$ are linearly independent over the ring $\mathfrak{Z}_0(H) = k[e^p, f^p, h^p - h, x^p, y^p]$ (which by itself is a polynomial ring in five variables). Indeed, let $0 = \sum_{i<p} a_i t^i$, where $a_i \in \mathfrak{Z}_0(H)$. We may assume that the elements a_i are homogeneous in x, y and the whole expression in the summation has the same degree n (in x, y). Thus we have $n = \deg(a_i) + 2i$, and since $\deg(a_i)$ are multiples of p, we have $2i = n \bmod(p)$. We deduce that at most one $a_i \neq 0$, hence all $a_i = 0$.

Now we claim that

$$t^p = e^p\big(x^p\big)^2 - f^p\big(y^p\big)^2 + \big(h^p - h\big)x^p y^p.$$

Indeed, let us consider the following central element $a = t^p - (e^p(x^p)^2 - f^p(x^p)^2)$. It is clear that this element is divisible by both x, y from the right, therefore from the argument above we see that it must be divisible by $x^p y^p$. Since a has degree $2p$, we must have

$$t^p = e^p\big(x^p\big)^2 - f^p\big(x^p\big)^2 + \omega x^p y^p,$$

where ω is some element in $k[e^p, h^p - h, f^p]$. Now let us consider the enveloping algebra filtration on H, using the well-known fact that $(a+b)^p = a^p + b^p \bmod A/[A, A]$ for any algebra A over k and any elements $a, b \in A$, we see that $t^p - (e^p(x^p)^2 - f^p(x^p)^2 - (h^p - h)x^p y^p)$ has the filtration degree less than $3p$. Hence, we see that

$$t^p - \big(e^p\big(x^p\big)^2 - f^p\big(x^p\big)^2 + \big(h^p - h\big)x^p y^p\big) = \alpha x^p y^p$$

where α is some constant. We claim that this constant must be 0. Indeed, if we write t^p as the sum of PBW monomials (x, y on the right), then it becomes clear that each monomial term appearing in t^p belongs to $\mathfrak{g}H$. Thus we may conclude that $\alpha = 0$. To conclude, we have shown that $\mathfrak{Z}(H)$ is isomorphic to $k[x_1, x_2, x_3, x_4, x_5, y]/(y^p - x_1x_2^2 - x_3x_4^2 + x_5x_2x_4)$, thus $\operatorname{Spec}(\mathfrak{Z}(H))$ is a ramified degree p covering of $\mathbb{A}^5$.

Now let us consider the case of an arbitrary z. We will show that elements $1, t_z, \ldots, t_z^{p-1}$ are linearly independent over $\mathfrak{Z}_0(H_z) = k[e^p, f^p, h^p, x_p, y_p]$ and generate $\mathfrak{Z}(H_z)$ as a module over $\mathfrak{Z}_0(H_z)$. This immediately follows from the corresponding statement for $1, t, \ldots, t^{p-1}$ over $\operatorname{gr}(\mathfrak{Z}(H_z))$. Thus there exists a monic polynomial $f(\tau)$ in one variable τ of degree p such that $\mathfrak{Z}(H_z)$ is isomorphic to $\mathfrak{Z}_0(H_z)[\tau]/(f(\tau))$. Thus $\operatorname{Spec}(\mathfrak{Z}(H_z))$ is a degree p covering of $\operatorname{Spec}(\mathfrak{Z}_0(H_z)) = \mathbb{A}^5$.

Let us write an explicit formula for the central element x_p when the parameter z is at most linear. Clearly, if z is a constant, then already x^p is central, thus we assume that $z = \Delta + a$, $a \in k$. We have

$$[z, x] = (2h - 3)x + 4ey,$$

so

$$\big[[z, x], x\big] = (\operatorname{ad} x)^2(z) = 2x^2 - 4ez,$$

hence

$$(\operatorname{ad} x)^{2n}(z) = (-4e)^{n-1}\big(2x^2 - 4ez\big)$$

and

$$\begin{aligned}(\operatorname{ad} x)^{2n+1}(z) &= \big[x, (\operatorname{ad} x)^{2n}(z)\big] = (-4e)^{n-1}\big(4e(2h-3)x + 4ey\big)\\ &= (-1)^{n-1}(4e)^n\big((2h-3)x + 4ey\big).\end{aligned}$$

Let us write p as $2k + 1$. We will make the following computation:

$$\big[f, e^k x\big] = e^k y - khe^{k-1}x + k(k-1)e^{k-1}x.$$

We claim that $[f, x^p + (-4)^k e^k x] = 0$. Indeed

$$\big[f, x^p\big] = -\big[x^p, f\big] = -(\operatorname{ad} x)^p(f) = (\operatorname{ad} x)^{p-2}(z).$$

Thus

$$\big[f, x^p\big] = (-1)^k (4e)^{k-1}\big((2h-3)x + 4ey\big),$$

and

$$\big[f, -(-4)^k e^k x\big] = -4^k e^k y + 4^k khe^{k-1}x(k-1)ke^{k-1}x = \big[f, x^p\big].$$

As a result, we get that $x_p = x^p + (-4)^k e^k x$ lies in the center of H_z.

Corollary 2.1. *Algebra H_z is a prime Noetherian ring which is Auslander regular and Cohen–Macaulay, and it is finite-dimensional over its center and its PI-degree is equal to p^2*

Proof. After passing to the associated graded and applying results of [BG], the only thing which needs to be checked is the part about the center. It is well known that H being the enveloping algebra of a restricted Lie algebra is a free module of dimension p^5 over $\mathfrak{Z}_0(H)$. Hence again by the associated graded argument, H_z is a free module of dimension p^5 over $\mathfrak{Z}_0$. Since $[K(\mathfrak{Z}(H_z)) : K(\mathfrak{Z}_0)] = p$ where $K(-)$ denotes the field of fractions, we see that the PI-degree of H_z is equal to p^2. $\square$

Now we turn to the study of irreducible representations of H_z. The main result is the following

Theorem 2.2. *For any $z \in k[\Delta]$ such that $\deg(z) < p - 1$, the Azumaya and the smooth loci of H_z coincide.*

Proof. By the previous result, H_z has PI-degree equal to p^2. Hence by the results of [BG], it suffices to show that there exists an open set $U \subset \operatorname{Spec}(\mathfrak{Z}(H_z))$ whose complement has codimension at least 2 in $\mathfrak{Z}(H_z)$, such that for any $\chi \in U$, algebra H_z has an irreducible representation of dimension $\geqslant p^2$.

Let us consider a set of characters $\chi \in \operatorname{Spec}\mathfrak{Z}(H_z)$ for which $\chi(e^p) = \chi(f^p) = 0$, $\chi(h^p - h) \neq 0$ thus $\mathfrak{sl}_2$ part is regular semi-simple, and $\chi(x_p) \neq 0$ or $\chi(y_p) \neq 0$. We claim that any such χ is in the Azumaya locus. Indeed, we may assume without loss of generality that $\chi(x_p) \neq 0$. Let V_χ be an irreducible module for χ. Since h acts diagonalizably and e acts nilpotently on V_χ, there exists a nonzero element $v \in V_\chi$ such that $ev = 0$, $hv = \lambda v$ for some $\lambda \in k$ (evidently $\lambda^p - \lambda = \chi(h^p - h)$.) Then, clearly $ex^i v = 0, h(x^i v) = (i + \lambda)x^i v$ for any $i = 0, \dots, p - 1$. It follows from the description of x_p that $x^p v = x_p v = \chi(x_p)v \neq 0$, in particular all x_i are nonzero. Therefore V_χ considered as a module over $\mathfrak{sl}_2$ has $V(\lambda + i)$ as a submodule, for all $i = 0, \dots, p - 1$ (where $V(\lambda + i)$ is an irreducible module over $\mathfrak{sl}_2$ for the regular semi-simple character χ corresponding to weight $\lambda + i$ [FP]). Thus, $\dim V_\chi \geqslant p^2$, hence χ is in the Azumaya locus.

Next let us consider characters χ with the property $\chi(e^p) = \chi(x_p) = \chi(h^p - h) = 0$. Let us denote by H_χ the fiber of H_z at the point $\chi \in \operatorname{Spec}(\mathfrak{Z}(H_z))$. Let us consider the module $M_\chi = H_\chi \otimes_{B_\chi} k_\chi$ (analog of the baby Verma module), where $k_\chi = kv$, $xv = ev = 0$, $hv = 0$, and B_χ is a subalgebra of H_χ generated by e, x, h. It is clear that $\dim M_\chi = p^2$ and M_χ is spanned by elements $f^i y^j v$, $i, j < p$. Our goal will be to produce $\beta^p = \chi(y_p)$ such that M_χ is irreducible. Let us assume that it is not irreducible. Let us choose a homogeneous weight element (with respect to $\operatorname{ad}(h)$) $g \in k[f, y]$ of smallest total degree in f, y such that $H_z gv$ is a proper submodule of M_χ. It is clear that we may choose such g with the property that $egv = xgv = 0$. It is also clear that y is invertible on M_χ. Therefore we may write

$$g = \sum_{i=0}^{n} a_i f^i y^{2n-2i}$$

where $a_0 = 1$, $a_n \neq 0$, $a_i \in k$, $2n < p$. It is easy to see that such a_i are determined uniquely (it follows from $[x, g]v = 0$), in particular, they do not depend on β. Let us write y_p as $y_p = y^p + fy^{p-2}b_0 + \cdots + f^k y b_k$, where $k = \frac{1}{2}(p - 1)$ and $b_i \in k[\Delta]$. Then we have the equality $y^p - c_0 fy^{p-2} - \cdots - \beta^p = 0$ in the endomorphism ring of M_χ. Let $L_\chi = M_\chi / H_z gv$, then $L(\chi) = k[f, y]/I$ where $I = \operatorname{Ann}(L_\chi) \cap k[f, y]$. We have that $f^p - 1$, $g \in I$, clearly we may choose β so that $y^p + c_0 fy^{p-2} + \cdots + \beta^p$ is invertible in $k[f, y]/I$ thus $L_\chi = 0$, a contradiction. Thus M_χ is irreducible for generic values of β.

Let us summarize: Let U be the SL_2-orbit (under the adjoint action) of characters χ such that $\chi(e^p) = \chi(f^p) = 0$, $\chi(h^p - h) \neq 0$ and either $\chi(x_p)$ or $\chi(y_p)$ is nonzero. Also, let V be an SL_2-orbit of characters χ such that $\chi(f^p) = 1$, $\chi(h^p - h) = 0$, $\chi(e^p) = 0$. Then we proved that U is in the Azumaya locus, and the Azumaya locus has a nonempty intersection with V. Now since V has codimension 1, and the complement of $U \cup V$ has codimension 2 (in $\operatorname{Spec}(\mathfrak{Z}(H_z))$), we may deduce that the complement of the Azumaya locus has codimension $\geqslant 2$. Hence, by the results of [BG], we are done. $\square$

In what follows we explicitly describe all irreducible modules for the case $z = 0$. Thus for a given central character χ, we want to describe corresponding irreducible modules of H_χ. We may assume without loss of generality that a central character χ satisfies the equality $\chi(e^p) = 0$ (using $SL_2(k)$-action). We will adopt the following notation. By N_χ we denote a subalgebra of H_χ generated by f, y and k_χ will denote its one-dimensional representation $kv = k_\chi$ such that $f^p v = \chi(f^p)v$, $y^p v = \chi(y^p)v$.

Proposition 2.3. *Let $\chi \in \operatorname{Spec}(\mathfrak{Z}(H_z))$ be a character such that $\chi(x^p) \neq 0$, then the module $M_\chi = H_\chi \otimes_{N_\chi} k_\chi$ is the irreducible rank p^2 module over H_χ. If $\chi(x^p) = 0$ and $\chi(y^p) \neq 0$ then $M_\chi = H_\chi \otimes_{B_\chi} k_\chi$ is the irreducible module of rank p^2. If $\chi(x^p) = \chi(y^p) = 0$ then any irreducible H_χ modules is an irreducible $\mathfrak{sl}_2$-module on which x, y act as 0.*

Proof. From the description of the center of H it follows that the singular locus of $\mathfrak{Z}(H)$ is $x^p = y^p = 0$. If $\chi \in \operatorname{Spec}(\mathfrak{Z}(H))$ belongs to the singular locus, and V_χ is an irreducible module of H_χ, then it is clear that there exists a nonzero $v \in V_\chi$, such that $xv = yv = 0$, which implies that $VV_\chi = 0$ (recall that $V = kx \oplus ky$). Thus H_χ is an irreducible $\mathfrak{sl}_2$-module.

Let a character χ be in the smooth locus, thus by the previous theorem, it belongs to the Azumaya locus. Hence we only need to show that modules in the proposition are nonzero and have dimension $\leqslant p^2$.

Let us start with the case when $\chi(x^p) = 0$. It is clear that dim $(M_\chi) = p^2$, so there is nothing to prove. Next, we consider the case when $\chi(y^p) = 0$. Thus M_χ is spanned by elements of the form $e^i x^j h^l v$. But we claim that $M_\chi = k[e, x]v$. Indeed, since $t = ey^2 + hxy - fx^2$ acts as a scalar, it follows that multiplication by hxy preserves $k[e, x]v$, so if $\chi(y^p) \neq 0$ then $hv \in k[e, x]v$. If $\chi(y^p) = 0$, then we have $hxyex^{p-2}v = -hx^p v$. Thus, $hv \in k[e, x]v$, so $M_\chi = k[e, x]v$, in particular $\dim M_\chi \leqslant p^2$. Now let us assume that $\chi(y^p) \neq 0$. Then from the fact that $hxyv \in k[e, y]v$ we see that $hv \in k[e, y]$, hence $M_\chi = k[e, x]v$. It is clear that an irreducible module over H_χ is a quotient of M_χ, thus $V_\chi = M_\chi$ and we are done. □

Acknowledgments

We would like to thank V. Ginzburg for suggesting the question, and A. Premet, I. Gordon for useful remarks.

References

[BC] K. Brown, K. Changtong, Symplectic reflection algebras in positive characteristic, arXiv: 0709.2338, 2007.
[BG] K. Brown, I. Goodearl, Homological aspects of Noetherian PI Hopf algebras and irreducible modules of maximal dimension, J. Algebra 198 (1997) 240–265.
[BGF] R. Bezrukavnikov, V. Ginzburg, M. Finkelberg, Cherednik algebras and Hilbert schemes in characteristic p, Represent. Theory 10 (2006) 254–298.
[FP] E. Friedlander, B. Parshall, Modular representation theory of Lie algebras, Amer. J. Math. 110 (6) (1988) 1055–1093.
[EGG] P. Etingof, W.L. Gan, V. Ginzburg, Continuous Hecke algebras, Transform. Groups 10 (3–4) (2005) 423–447.
[Kh] A. Khare, Category $\mathcal{O}$ over a deformation of the symplectic oscillator algebra, J. Pure Appl. Algebra 195 (2) (2005) 131–166.
[St] R. Steinberg, Lectures on Chevalley Groups, Yale University, New Haven, CT, 1968.
[Ti] A. Tikaradze, Center of infinitesimal Hecke algebra of $\mathfrak{sl}_2$, arXiv: math.QA/0604498, 2006.
[V] F. Veldkamp, The center of the universal enveloping algebra of a Lie algebra in characteristic p, Ann. Sci. Ec. Norm. 5 (4) (1972) 217–240.